OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2615**

Statistics 3

Monday **21 JANUARY 2002** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 The wages department of a large company models the incomes of the employees by the continuous random variable X with cumulative distribution function

$$F(x) = 1 - \left(\frac{3}{x}\right)^4, \qquad 3 \leqslant x < \infty,$$

where X is measured in an arbitrary currency unit. (X is said to have a Pareto distribution.)

(i) Find the median income. Find also the smallest income of an employee in the top 10% of incomes. [5]

(ii) Find the probability density function of X and hence show that the mean income is 4. [5]

(iii) Find the probability that a randomly chosen employee earns more than the mean income. [2]

(iv) Given that the variance of X is 2, use a Normal approximation to find the probability that the mean income of a random sample of 40 employees exceeds the median of X. [3]

2 Sugar is automatically packed by a machine into bags of nominal weight 1000 g. Due to random fluctuations and the set-up of the machine, the weights of bags are in fact Normally distributed with mean 1020 g and standard deviation 25 g. Two bags are selected at random.

(i) Find the probability that the total weight of the two bags is less than 2000 g. [4]

(ii) Find the probability that the weights of the two bags differ by less than 20 g. [4]

Another machine is also in use for packing sugar into bags of nominal weight 1000 g. It is assumed that the distribution of the weights for this machine is also Normal. A random sample of 9 bags packed by this machine is found to have the following weights (in g).

1012	996	984	1005	1008	994	1003	1017	1002

(iii) Test at the 5% level of significance whether it may be assumed that the mean weight for this machine is 1000 g. [7]

3 The amounts of money spent by customers on meals at a certain restaurant are Normally distributed with mean and standard deviation (in pence) 1424 and 108 respectively. The manager introduces new menus in the hope of increasing the amount spent. It is assumed that the standard deviation remains the same.

After the introduction of the new menus, the amounts spent by a random sample of 20 customers are found to have mean 1465 pence.

(i) Test at the 5% level of significance whether the mean amount spent by customers has increased. [6]

(ii) Provide a two-sided 95% confidence interval for the mean amount now spent by customers. [4]

(iii) Calculate the probability of a Type II error for the test in part (i) if the true value of the mean amount now spent by customers is 1500 pence. What does this probability represent? [5]

4 As part of a survey, a railway company took a sample of 80 people, each of whom had recently travelled 3 times on a particular route, and asked them on how many of these 3 occasions they were generally satisfied with their journeys. The results were as follows.

Number of occasions generally satisfied	0	1	2	3
Number of people	4	20	44	12

The company is considering fitting a binomial model to these data, with p taken as the probability of being generally satisfied on a journey.

(i) Estimate the value of p. [2]

(ii) Use a suitable statistical procedure and a 10% significance level to assess the goodness of fit of a binomial model. [10]

(iii) What assumption is required about the sample? Discuss briefly whether this assumption is likely to hold. [3]

Mark Scheme

1(i)	$F(x)=1-\left(\frac{3}{x}\right)^4,\ x\ge 3$	Condone sloppy notation (e.g. omission of dx) throughout.	
	Median m: $\frac{1}{2}=1-\left(\frac{3}{m}\right)^4$	M1 Set up equation $F(m)=\frac{1}{2}$.	
	$\therefore\left(\frac{3}{m}\right)^4=\frac{1}{2}$	M1 Attempt to solve. Previous M1 may be implied by correct work here.	
	$\therefore m^4=2\times 3^4,\ m=3\times\sqrt[4]{2}=\underline{\underline{3\cdot567(62)}}$	A1 c.a.o. 2 dp or better. Accept surds.	
	Want k s.t. $0\cdot9=1-\left(\frac{3}{k}\right)^4$	M1 Formulation of equation for 90th percentile.	
	$\therefore\left(\frac{3}{k}\right)^4=0\cdot1,\ k^4=\frac{3^4}{0\cdot1},$		
	$k=\frac{3}{\sqrt[4]{0\cdot1}}=\underline{\underline{5\cdot334(83)}}$	A1 c.a.o. 2 dp or better. Accept surds.	5
(ii)	P.d.f. $f(x)=\frac{d}{dx}F(x)$	M1 An attempt to differentiate.	
	$=-3^4(-4)x^{-5}=\underline{\underline{4\times3^4x^{-5}}}$	A1 Accept in unsimplified form.	
	$\mu=\int_3^\infty 4\times3^4x^{-4}dx$	M1 Integral of $x\times$c's $f(x)$, not including the limits. Condone omission of or errors with notation, e.g. "$E[X]$" or "μ".	
	$=4\times3^4\left[\frac{x^{-3}}{-3}\right]_3^\infty=4\times3^4\times\frac{1}{3}\times3^{-3}=\underline{\underline{4}}$	M1 Limits correct. A1 Convincingly shown. Beware printed answer.	5
(iii)	Want $\int_4^\infty 4\times3^4x^{-5}dx$	M1 Correct integral, including limits. Allow c's $f(x)$.	
	$=4\times3^4\left[\frac{x^{-4}}{-4}\right]_4^\infty=\frac{3^4}{4^4}=\underline{\underline{0\cdot316(41)}}$	A1 f.t. c's $f(x)$ provided answer lies between 0 and 1.	2
	[ALITER by cdf: $F(4)=1-\left(\frac{3}{4}\right)^4=0\cdot683(59)$	(M1)	
	$\therefore$ want 1 - this $=0\cdot316(41)$]	(A1)	
(iv)	$\bar{X}_{40}\sim$ approx $N\left(4,\frac{2}{40}\right)$	M1 For μ and σ^2/n stated or used. Must involve n correctly.	
	$=N(4, 0\cdot05\,[\sigma=0\cdot2236])$	M1 Formulating the problem as a 1-sided inequality and standardising.	
	$P(\bar{X}_{40}>3\cdot568)$ $=P\left(N(0,1)>\frac{-0\cdot432}{0\cdot2236}=-1\cdot932\right)$ $=\underline{\underline{0\cdot9733}}$	Allow f.t. of c's median and all parameters for this mark. Accept equivalent forms e.g. based on symmetry. Be lenient w.r.t. abuse of inequalities; diagrams may well provide helpful clarification.	
		A1 c.a.o. Accept a.w.r.t. 0·973 Accept alternative method using $T=X_1+\ldots+X_{40}\sim N(160, 80)$ to find $P(T>142\cdot72)$.	3
			15

2(i)	$X \sim N(1020, \sigma = 25\,[\sigma^2 = 625])$ $X_1 + X_2 \sim N(2040, 625 + 625 = 1250)$ $P(X_1 + X_2 < 2000)$ $= P\left(N(0,1) < \frac{-40}{\sqrt{1250}} = -1\cdot131\right)$ $= 1 - 0\cdot8710 = \underline{\underline{0\cdot1290}}$	B1 Mean. B1 Variance (or σ provided it's clear) M1 Formulation and standardising using c's μ and σ. A1 f.t. c's μ and σ.	4
(ii)	$X_1 - X_2 \sim N(0, 625 + 625 = 1250)$ $P(-20 < X_1 - X_2 < 20)$ $= P\left(\frac{-20}{\sqrt{1250}} < N(0,1) < \frac{20}{\sqrt{1250}}\right)$ $= P(-0\cdot5657 < N(0,1) < 0\cdot5657)$ $= 2 \times 0\cdot2142 = \underline{\underline{0\cdot4284}}$	B1 Mean. B1 Variance (or σ provided it's clear). M1 Formulation and standardising as a 2-sided inequality using c's μ and σ (?0). Do not allow if only $X_1 - X_2 < 20$ used. A1 f.t. c's μ and σ.	4
(iii)	t-test of H_0: μ=1000 against H_1: μ?1000 $\bar{x} = 1002\cdot33$ $s_{n-1}^2 = 99\cdot25,\ s_{n-1} = 9\cdot9624$ Test statistic is $t = \frac{1002\cdot33 - 1000}{\frac{9\cdot9624}{\sqrt{9}}}$ $= \underline{\underline{0\cdot7026}}$ Refer to t_8 Double tailed 5% point is 2·306 Not significant. May assume mean weight is now 1000.	B1 Both $\bar{x}$ and s_{n-1}^2 seen at some point. Allow $s_n^2 = 88\cdot2222,\ s_n = 9\cdot3927$ but ONLY if correctly used in sequel. M1 Accept c's $\bar{x}$ and s_{n-1}. Numerator might be given as $\mu - \bar{x}$; allow M1A0 for this, then f.t. subsequently. Allow $s_n/\sqrt{8}$ (see above). A1 c.a.o. Correct answer ww scores 2/2 and may imply B1 above also. f.t. from here if incorrect. Exact value depends on accuracy of values used (e.g. $\bar{x} = 1002\cdot33$ gives 0·7016). M1 May be awarded even if test statistic is wrong. Must see evidence of intention to use t-distribution. But no f.t. if ν is wrong. A1 No f.t. if wrong. May be +ve or –ve. B1 For comparison (p.i.) and simple conclusion (p.i.) consistent with c's t and critical value. B1 Consistent contextual conclusion, including an indication that it is the *mean* weight. SC t_9 and 2·262, or t_8 and 1·860, used can score max B1 for either form of conclusion seen.	7
			15

3(i)	Test of H_0: μ=1424 against H_1: μ>1424		
	Test statistic is $z = \dfrac{1465-1424}{\frac{108}{\sqrt{20}}}$	M1 Numerator might be given as 1424−1465. Allow M1A0 then f.t. for $\mu-\bar{x}$ in numerator.	
	$= \underline{1.697(76)}$	A1 c.a.o. Correct answer ww scores 2/2. f.t. from here if incorrect.	
	Refer to N(0, 1).	M1 May be awarded even if test statistic is wrong. Must see evidence of intention to use Normal distribution. But no f.t. if wrong.	
	Upper 5% point is 1·645	A1 No f.t. if wrong. Must be +ve.	
	Significant.	B1 For comparison (p.i.) and simple conclusion (p.i.) consistent with c's z and critical value.	
	Seems mean amount spent has increased.	B1 Consistent contextual conclusion, including an indication that it is the *mean* amount.	6
		SC 1·96 used can score max B1 for either form of conclusion seen.	
(ii)		N.B. ZERO OUT OF 4 if not same distribution as used for test. Same wrong distribution can score max M1B0M1A0. BUT allow recovery to Normal for possible 4/4.	
	95% C.I. given by:	M1 for 1465 ± ...	
	$1465 \pm 1.96 \times \dfrac{108}{\sqrt{20}} = 1465 \pm 47.33(31)$	B1 for 1·96 (from N(0, 1)). M1 for 108/√20.	
	$= (\underline{\underline{1417.66(69)}}, \underline{\underline{1512.33(31)}})$	A1 BOTH. c.a.o. Accept correct ww for 4/4. Must be an interval. Min 1 dp required.	4
(iii)	If μ = 1500, $\bar{X} \sim N\left(1500, \dfrac{108^2}{20}\right)$	M1 Distribution of $\bar{X}$ with new mean.	
	H_0 is accepted if $\bar{X} < 1424 + 1.645 \times \dfrac{108}{\sqrt{20}} = 1463.73$	M1 Find the critical value of the original test. Just 1424 or 1465 is not acceptable.	
	So $P(\text{Type II err}) = P\left(N\left(1500, \dfrac{108^2}{20}\right) < 1463.73\right)$ $= P(N(0,1) < -1.502)$ $= \underline{\underline{0.0666}}$	M1 Formulate the condition for a Type II error by combining the above, but not when c.v. is 1424 or 1465. A correct statement here could imply the previous 2 marks. A1 c.a.o.	
	This is the probability of wrongly accepting the null hypothesis μ = 1424 when in fact μ = 1500 (where μ is the mean amount now spent).	E1 Explain in context. Must include at least a reference to the fact that μ = 1500.	5
			15

5515 Version of Question 3

3(i)	Test of H_0: μ=1424 against H_1: μ>1424		
	Test statistic is $z = \dfrac{1465-1424}{\frac{108}{\sqrt{20}}}$	M1 Numerator might be given as 1424−1465. Allow M1A0 then f.t. for $\mu - \bar{x}$ in numerator.	
	$= \underline{1.697(76)}$	A1 c.a.o. Correct answer ww scores 2/2. f.t. from here if incorrect.	
	Refer to N(0, 1).	M1 May be awarded even if test statistic is wrong. Must see evidence of intention to use Normal distribution. But no f.t. if wrong.	
	Upper 5% point is 1·645	A1 No f.t. if wrong. Must be +ve.	
	Significant.	B1 For comparison (p.i.) and simple conclusion (p.i.) consistent with c's z and critical value.	
	Seems mean amount spent has increased.	B1 Consistent contextual conclusion, including an indication that it is the *mean* amount.	6
		SC 1·96 used can score max B1 for either form of conclusion seen.	
(ii)		N.B. ZERO OUT OF 4 if not same distribution as used for test. Same wrong distribution can score max M1B0M1A0. BUT allow recovery to Normal for possible 4/4.	
	95% C.I. given by:	M1 for 1465 ± ...	
	$1465 \pm 1 \cdot 96 \times \dfrac{108}{\sqrt{20}} = 1465 \pm 47 \cdot 33(31)$	B1 for 1·96 (from N(0, 1)).	
		M1 for 108/√20.	
	$= (\underline{\underline{1417 \cdot 66(69)}}, \underline{\underline{1512 \cdot 33(31)}})$	A1 BOTH. c.a.o. Accept correct ww for 4/4. Must be an interval. Min 1 dp required.	4
(iii)	"Business" and "private" might well be different.	E1	
	A random sample might be biased one way or the other ...	E1	
	... and therefore unrepresentative.	E1	
	Stratified sampling ...	E2 Accept a description of (something like) stratified sampling (not necessarily known by name).	5
			15

4(i)	Number of occasions: 0, 1, 2, 3 Number of people: 4, 20, 44, 12 $\Sigma = 80$		
	$\hat{p} = \dfrac{0\times4+1\times20+2\times44+3\times12}{240} = \dfrac{144}{240} = \underline{\underline{0\cdot6}}$	M1 Correct complete expression. A1 c.a.o.	2
(ii)	Consider B(3, 0·6)	M1 Expected frequencies. f.t. c's estimate of p provided $0<p<1$. −1 e.e., but 2 errors	

x	0	1	2	3
P(X=x)	0.064	0.288	0.432	0.216
Obs f	4	20	44	12
Exp f	5.12	23.04	34.56	17.28

A1 and sum = 80 counts as only 1 error. Allow max M1if e's do not add up to 80. Award A0 if e's rounded to integers. f.t. from here provided sum = 80.

	$X^2 = 0{\cdot}245 + 0{\cdot}4011 + 2{\cdot}5785 + 1{\cdot}6133$ $= \underline{\underline{4{\cdot}84}}$	M1 f.t. from incorrect e's (including when cells A1 are combined as a consequence). f.t. from here if incorrect.	
	Refer to χ^2_2.	M1 Can be earned even if X^2 is wrong. f.t. ν = no. cells used − 2, but not when n = 80 or 240. Accept anything which implies the use of χ^2_2, including LH tail.	
	Upper 10% point is 4·605.	A1F f.t. correct ν from c's table.	
	Significant.	B1 For comparison (p.i.) and preliminary conclusion (p.i.) consistent with c's X^2 and critical value. Accept equivalents e.g. "Accept H_1".	
	So there is some evidence that the model does not fit.	B1 For a consistent contextual conclusion. SC χ^2_3 and 6·251 used can score max B1 for either form of conclusion seen.	
	It appears to fit reasonably well for 0 or 1 satisfactory journeys, ... but under-predicts for 2 and over-predicts for 3.	E2 If not awarded here then allow recovery of these marks for comments in (iii) about the likelihood of meeting the conditions for the Binomial model.	10
(iii)	Sample should be random.	E1	
	It is difficult to see how this could be achieved for people who have recently made (exactly) 3 journeys. More likely to be a quota sample?	E2 Reward any two sensible, relevant comments or points regarding selection of the sample.	3
			15

Examiner’s Report

Statistics 3 (2615)

General Comments

For this component, the overall standard of the candidates was generally pleasing: many were clearly well prepared for it. There were far fewer problems than in the past with a number of key issues; these are discussed in the comments on individual questions below. However, as in the past, candidates were less successful at providing appropriate comments or interpretation. Their explanations, often woolly, indicated that they had not understood what they were being asked to consider. In addition there are some candidates who do not take sufficient care over the accuracy of their calculations. For example, they appear not to realise that early rounding can lead to substantially different outcomes.

Invariably all four questions were attempted. Questions 1 and 2 were found to be particularly accessible and proved to be high scoring for well-prepared candidates. In Question 3 there were problems with part (iii). By the time they reached Question 4 some candidates appeared to be running out of time and many careless errors were made, possibly as a result of rushing.

Comments on Individual Questions

Question 1 (Continuous random variables; employee's incomes.)

This question was found to be very accessible; almost one third of the candidates scored full marks for it.

(i) There were many false starts (due to settling of nerves, perhaps) but most candidates went on to do this part well, finding both the median and 90th percentile easily. There were a few candidates who did not recognise that they had been given F(x) and so thought that they should integrate to find the median.

(ii) Most candidates knew what to do here but some were let down by their inability to differentiate correctly in order to obtain the p.d.f. This then meant that these candidates were unable to obtain the stated mean satisfactorily.

(iii) This part was usually correct. It was pleasing that so many candidates realised that they could use the c.d.f. to answer it directly.

(iv) Compared with previous years there was a marked improvement in the candidates' treatment of this type of question. There was much more certainty about the mean and variance of the sample mean and far less confusion over whether to use $\overline{X}$ or ΣX, with most preferring the former.

[(ii) Median 3·568, 90th percentile 5·335; (ii) $\frac{324}{x^5}$; (iii) 0·3164; (iv) 0·9733]

Question 2 (Linear combinations of Normal variables and hypothesis test of the mean; weights of bags of sugar.)

(i) This part was almost always answered well.

(ii) Here the parameters for the difference of two bags of sugar were usually stated correctly. However, even strong candidates overlooked the need to consider a two-sided inequality, and so ended up finding only $P(X_1 + X_2 < 20)$.

(iii) There were many fully correct answers to this part. The work seen represented an encouraging improvement compared to the responses to recent similar questions. There were fewer instances of candidates using s_n instead of s_{n-1} to calculate the test statistic, and candidates were more likely to identify the correct t distribution and critical value than in the past.

[(i) 0·1290; (ii) 0·4284; (iii) test statistic 0.7026, critical value 2·306]

Question 3 (Hypothesis test and confidence interval for population mean from a small sample; Type II error; amounts spent in a restaurant.)

(i) As in Question 2 part (iii), it was noticeable that candidates were much clearer about what they needed to do: their ability to identify the correct distribution and critical value was much improved. Consequently most candidates did well in this part. There was a smattering of the usual errors such as forgetting to divide the variance by the sample size, or using the wrong percentage point from the Normal tables.

(ii) The candidates' attempts to find the confidence interval benefited from the improvements identified above. Once again errors similar to those referred to in part (i) were seen from time to time.

(iii) This part of the question posed some difficulty; the topic "Type I and Type II errors" is new to Statistics 3. However, it was tested last summer in the first paper of the new specification. It was evident that many candidates did not understand what they needed to do, and it seemed that the level of preparation varied from centre to centre. It was necessary for candidates to find the critical value of the original test in part (i), to recognise that there was a new distribution, with mean 1500, for the sample means, and to put these together to find the required probability. Finally they were expected to interpret this probability *in context*, which they rarely did satisfactorily.

[(i) Test statistic 1·698, critical value 1·645; (ii) (1417·67, 1512·33); (iii) 0·0666]

Question 4 (Chi-squared test for goodness of fit of a Binomial model; customer satisfaction with railway journeys.)

(i) The correct estimate of p was often seen, but it was as common to see an incorrect estimate. All sorts of different strategies were employed to produce a variety of incorrect values. The usual approach was to find the mean of the given data and then divide by n, but there was a certain amount of confusion about what was the correct value of n.

(ii) Whatever the value of p obtained, candidates went on to calculate correctly the expected frequencies for their distribution. For most the calculation of X^2, the test statistic, also followed correctly. However, depending on the value of p used, some candidates inadvertently produced a situation where it would have been appropriate to combine frequencies, and those concerned did not always notice. In contrast with earlier questions, the identification of the number of degrees of freedom and the critical value was not as reliable. In asking candidates to "assess the goodness of fit of the binomial model" it was intended that their responses would contain rather more detail than a simple conclusion to the hypothesis test. This detail was conspicuously absent from their answers.

(iii) In this part, the question invited candidates to consider the nature of the *sample* which would be required for the test, to reflect on whether or not such a sample was likely and, possibly, to suggest what might have actually happened. Instead almost all candidates chose to discuss issues relating to the validity of the binomial model: the independence of events and whether p could be considered to be constant.

[(i) 0.6; (ii) Expected frequencies 5·12, 23·04, 34·56, 17·28, $X^2 = 4·84$; $\nu = 2$, critical value 4·605]